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THE STABILITY OF MOTION OF A SYSTEM WITH AFTEREFFECT†

I. V. BOIKOV

Penza

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A system with aftereffect is considered. The state of the system at any instant of time t depends not only on its phase coordinates at the instant t but also on the phase coordinates at the preceding instants of time $[\gamma_i(t), t]$, where $\gamma_i(t) \leqslant t$, $i = 1, 2, \ldots, n$ (in the special case when $\gamma_i(t) \equiv t_0$ for $i = 1, 2, \ldots, n$). The stability of such systems is investigated using Lyapunov's second method.

We will investigate the stability of motion corresponding to the zeroth solution of the equation

$$\frac{d\mathbf{x}}{dt} = A(t)\mathbf{x} + \int\limits_{\Gamma(t)}^{t} \mathbf{K}(t, s, \mathbf{x}(s))ds + \mathbf{F}(\mathbf{x}, \mathbf{u}, t), \quad \mathbf{u} \in R^n \tag{1}$$

where $\gamma_i(t) \leqslant t$, $\gamma_i(0) = \beta_0$ the matrix $A(t) = \{a_{ij}(t)\}$ and the vector function $\Gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t))$, $\mathbf{K}(t, s, \mathbf{x}(s)) = (K_1(t, s, \mathbf{x}(s)), \ldots, K_n(t, s, \mathbf{x}(s)))$, $\mathbf{F}(\mathbf{x}, \mathbf{u}, t) = (f_1(\mathbf{x}, \mathbf{u}, t), \ldots, f_n(\mathbf{x}, \mathbf{u}, t))$ are continuous in the region $\Omega_1 \times \Omega_2 \times [0, \infty)$; $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{u} = (u_1, \ldots, u_n)$; $F(0, \mathbf{u}, t) \equiv 0$.

Here Ω_1 is a certain neighbourhood of the point $\mathbf{x} = \mathbf{0}$, Ω_2 is the region in which the vector $\mathbf{u}(t) = (u_1(t), \ldots, u_n(t))$ is defined, and the subscripts i, j take values $1, 2, \ldots, n$.

Note that the case considered in [1], when the control vector $\mathbf{u}(t)$ is specified by analytic functionals, represented by absolutely converging Volterra–Frechet series

$$u_i = \sum_{k=1}^{\infty} \sum_{j_1 \ldots j_k=1}^{n} \int\limits_0^t \ldots \int\limits_0^t K_i^{j(k)}(t, s_1, \ldots, s_k) x_{j_1}(s_1) \ldots x_{j_k}(s_k) ds_1, \ldots, dx_k$$

where $K_i^{j(k)}(t, s_1, \ldots, s_k)$ are continuous functions specified in the set $J_k' = \{(t, s_1, \ldots, s_k) \in R^{k+1}$, $0 \leqslant s_r \leqslant t \leqslant \infty$, $r = 1, 2, \ldots, k\}$, is also included in the formula specifying the control vector described above.

Equations of the form (1) find application in problems of viscoelasticity [2, 3], aeroelasticity [4], and also when investigating economic models [5].

When $\gamma_i(t) \equiv t_0$, the analytic vector function $\mathbf{F}(\mathbf{x}, \mathbf{u}, t)$ and the stability of Eq. (1) with a linear integral operator were investigated in [1] using Lyapunov's first method.

The stability of the equations with aftereffect has been considered in many publications [6–8]. A method of investigating the stability of systems of differential equations in the critical case when there is a single zero root was proposed in [9, 10], based on an investigation of the spectrum of the Jacobian on the right-hand side of the equation in the neighbourhood of the perturbed solution. This method was extended in [11, 12] to differential and difference equations and to all kinds of critical cases.

Below it is extended to systems described by equations of the form (1).

Equation (1) will be investigated in the n-dimensional space R_n. We can take one of the following as the norm in R_n

$$\|\mathbf{x}\|_1 = \left[\sum_{k=1}^{n} |x_k|^2\right]^{1/2}, \quad \|\mathbf{x}\|_2 = \max|x_i|, \quad \|\mathbf{x}\|_3 = \sum_{k=1}^{N} |x_k|$$

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We will use the following notation below: $R(\mathbf{a}, r) = \{\mathbf{x} \in R_n : || \mathbf{x} - \mathbf{a} || \leq r\}$, $S(\mathbf{a}, r) = \{\mathbf{x} \in R_n : || \mathbf{x} - \mathbf{a} || = r\}$, $\operatorname{Re} A = A_R = (A + A^*)/2$, ΛA is the logarithmic norm of the linear operator A, defined in [13] by the expression $\Lambda A = \lim_{h\downarrow 0}(|| I + hA || - 1)/h$.

Suppose $r > 0$. We will denote by $\phi(t)$ the arbitrary continuously differentiable curve $\phi(t) = (\phi_1(t), \ldots, \phi_n(t))$, which belongs to the sphere $R(\mathbf{0}, r)$. We will fix the arbitrary instant of time T.

We introduce the vector $C = \operatorname{col}(c_1, \ldots, c_n)$ where $c_i = \phi_i(T)$. We also introduce the following notation

$$B(\mathbf{C},T) = \{b_{ij}(\mathbf{C},T)\}, \quad H(\mathbf{C},T) = \{h_{ij}(\mathbf{C},T)\}$$

$$b_{ij}(\mathbf{C},T) = \begin{cases} x_i(T)/(mc_j), & c_j \neq 0 \\ 0, & c_j = 0 \end{cases}, \quad \chi_i(T) = \int_{\gamma_i(T)}^{T} K_i(T,\tau,\phi(\tau))d\tau$$

$$h_{ij}(\mathbf{C},T) = \begin{cases} [F_i(0,\ldots,0,c_j,c_{j+1},\ldots,c_n,u(T),T) - \\ -F_i(0,\ldots,0,0,c_{j+1},\ldots,c_n,u(T),T)]/c_j, & c_j \neq 0 \\ 0, & c_j = 0 \end{cases}$$

where m is the number of non-zero coordinates of the vector $\mathbf{C}$.

Theorem 1. Suppose, for any fixed value of t, $0 \leq t \leq \infty$, for any non-zero vector $\mathbf{C} \in R(\mathbf{0}, r)$, where $r > 0$, for any continuously differentiable curve $\phi(t) = (\phi_1(t), \ldots, \phi_n(t))$, belonging to the sphere $R(\mathbf{0}, r)$, and such that $\phi_i(t) = c_i$, the following condition is satisfied

$$\Lambda(A(t) + B(\mathbf{C}, t) + H(\mathbf{C}, T)) < 0, \qquad (\Lambda(A(t) + B(\mathbf{C}, t) + H(\mathbf{C}, T)) < -\alpha, \ \alpha = \text{const} > 0)$$

Then the trivial solution of Eq. (1) is stable (asymptotically stable).

Proof. We fix an arbitrary $r_1 < r$ and we will show that when the conditions of the theorem are satisfied each solution of Eq. (1), in which the values of the function $x(t)$ for $\beta_0 \leq t \leq 0$ are situated in the sphere $R(\mathbf{0}, r_1)$, does not leave this sphere. We will assume the opposite: suppose that at the instant of time $t = T$ the trajectory $x(t)$ of Eq. (1) leaves the sphere $R(\mathbf{0}, r_1)$.

We introduce the following notation

$$d_{il} = a_{il}(T) + \chi_i(T)/mx_l(T) + [F_i(0,\ldots,0,x_l(T),\ldots,x_n(T),\mathbf{u}(T),T) -$$
$$-F_i(0,\ldots,0,0,x_{l+1}(T),\ldots,x_n(T),\mathbf{u}(T),T)]/x_l(T)$$
$$g_i(t,\mathbf{x}(t)) = \sum_{l=1}^{n}(a_{il}(t) - a_{il}(T))x_l(t) - \sum_{l=1}^{n}\chi_i(T)\frac{(x_l(t) - x_l(T))}{mx_l(T)} -$$
$$-\frac{F_i(x_1(T),\ldots,x_n(T),\mathbf{u}(T),T) - F_i(0,x_2(T),\ldots,x_n(T),\mathbf{u}(T),T)}{x_1(T)}(x_1(t) - x_1(T)) -$$
$$-\ldots-\frac{F_i(0,\ldots,0,x_n(T),\mathbf{u}(T),T) - F_i(0,\ldots,0,u(T),T)}{x_n(T)}(x_n(t) - x_n(T)) +$$
$$+F_i(\mathbf{x}(t),\mathbf{u}(t),t) - F_i(\mathbf{x}(T),\mathbf{u}(T),T)$$

where m is the number of non-zero numbers among $x_1(T), \ldots, x_n(T)$.

Note that if for a certain l $(1 \leq l \leq n)$ $x_l(T) = 0$, then in the notation d_{il} and g_i terms in which the denominator is equal to $x_1(T)$, are omitted.

Then, when $t \geq T$ we can represent the system of equations (1) in the form

$$d\mathbf{x}/dt = D\mathbf{x} + \mathbf{G}(t,\mathbf{x}(t)), \quad D = \{d_{ij}\}, \ \mathbf{G}(t,\mathbf{x}(t)) = \operatorname{col}(g_1(t,\mathbf{x}(t)),\ldots,g_n(t,\mathbf{x}(t))) \tag{2}$$

The solution of Eq. (2) when $t \geq T$ has the form

$$\mathbf{x}(t) = e^{D(t-T)}\mathbf{x}(T) + \int_T^t e^{D(t-s)}\mathbf{G}(s,\mathbf{x}(s))ds \tag{3}$$

It follows from the structure of the operator $\mathbf{G}(t, \mathbf{x}(t))$ that for any $\varepsilon(\varepsilon > 0)$ as small as desired we can choose a value of Δt such that for $T \leqslant t \leqslant T + \Delta t$ $||\mathbf{G}(t, \mathbf{x}(t))|| \leqslant \varepsilon ||\mathbf{x}(t)||$.

Changing in Eq. (3) to norms, we have for $T \leqslant t \leqslant T + \Delta t$

$$||\mathbf{x}(t)|| \leq e^{\Lambda(D)(t-T)}\mathbf{x}(T) + \varepsilon \int_T^t e^{\Lambda(D)(t-s)} ||\mathbf{x}(s)|| ds \tag{4}$$

We introduce the new variable $\psi(t) = e^{-\Lambda(D)t} ||\mathbf{x}(t)||$. Inequality (4) then takes the form

$$\psi(t) \leq \psi(T) + \varepsilon \int_T^t \psi(s) ds \tag{5}$$

Applying the Gronwall–Bellman inequality to (5) and reverting to the norm $||\mathbf{x}(t)||$, it can be shown that for $T \leqslant t \leqslant T + \Delta t$ the following estimate holds

$$||\mathbf{x}(t)|| \leq \exp\left((\Lambda(D) + \varepsilon)(t - T)\right)||\mathbf{x}(T)||$$

Since by the conditions of the theorem the logarithmic norm is negative, then taking ε such that $\Lambda(D) + \varepsilon \leqslant 0$ and choosing from it the corresponding value of Δt^*, it can be shown that in the time interval $[T, T + \Delta t^*]$ the trajectory of the solution of Eq. (1) does not leave the sphere $S(\mathbf{0}, r_1)$. The stability of the solution of Eq. (1) follows from this contradiction.

We can similarly prove the asymptotic stability. The theorem is proved.

Note. If the system of equations (1) is investigated in Euclidean space E_n, then in formulating the theorem the condition

$$\Lambda(A(t) + B(C, t) + H(C, T)) < 0 (\Lambda(A(t) + B(C, t) + H(C, T)) < -\alpha)$$

can be replaced by the following

$$\sigma\left(\operatorname{Re}\left(A(t) + B(C, t) + H(C, t)\right)\right) < 0 \quad \sigma\left(\left(\operatorname{Re}\left(A(t) + B(C, t) + H(C, t)\right)\right) < -\alpha\right)$$

We will consider some classes of problems for which the conditions of Theorem 1 are easily verified. We will consider as one of these classes equations of the form (1) in which

$$\sup_{t_0 \leq t < \infty} \max_{i=1,\ldots,n} |t - \gamma_i(t)| \leq H \tag{6}$$

Suppose $r > 0$. We fix an arbitrary value of T. Suppose $\boldsymbol{\zeta}(T) = (\zeta_1(T), \ldots, \zeta_n(T))$, $\zeta_i(T) \in (\gamma_i(T), T)$. Suppose $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_n)$ is an arbitrary point situated inside the sphere $S(\mathbf{0}, r)$, while $\mathbf{s}(r) = (s_1(r), \ldots, s_n(r))$ is an arbitrary point situated on the sphere surface $S(\mathbf{0}, r)$. We introduce the following notation

$$D = \{d_{il}\}, \quad d_{il} = d_{il}(T, \zeta_i(T), \boldsymbol{\eta}, s(r)) = a_{il}(T) + K_i(T, \zeta_i(T), \boldsymbol{\eta})(T - \gamma_i(T))/(ms_l(r)) +$$
$$+ [F_i(0, \ldots, 0, s_l(r), \ldots, s_n(r), \mathbf{u}(T), T) - F_i(0, \ldots, 0, s_{l+1}(r), \ldots, s_n(r), \mathbf{u}(T), T)]/s_l(r)$$

where m is the number of non-zero coordinates of the vector $\mathbf{s}(r) = (s_1(r), \ldots, s_n(r))$.

Theorem 2. Suppose conditions (6) are satisfied. If for fairly small $0 < r \leqslant r^*$ and arbitrary $T(t_0 \leqslant T < \infty)$, $\boldsymbol{\zeta}(T)$, $\boldsymbol{\eta} \in R(0, r)$, $\mathbf{s}(r) \in S(\mathbf{0}, r)$ the logarithmic norm of the matrix D is negative (less than $-\alpha$, $\alpha > 0$), the solution of Eq. (1) is stable (asymptotically stable).

Proof. We will fix an arbitrary fairly small value of $r(0 < r \leqslant r^*)$ and show that when the conditions of the theorem are satisfied each solution of Eq. (1), in which the values of the function $x(t)$ for $\beta_0 \leqslant t \leqslant 0$ are situated in the sphere $R(\mathbf{0}, r)$, does not leave this sphere. We will assume the opposite: at the instant of time $t = T$ the trajectory $x(t)$ of Eq. (1) leaves the sphere $R(\mathbf{0}, r)$. We will denote the point of intersection of the trajectory $x(t)$ with the sphere $S(\mathbf{0}, r)$ by $\mathbf{s}(r) = (s_1(r), \ldots, s_n(r))$. Using the theorem on the mean we have $\chi_i(T) = K_i(T, \zeta_i(T), \eta)(T - \gamma_i(T))$.

Then the ith equation of the system of equations (1) can be represented in the form

$$\frac{dx_i}{dt} = \sum_{l=1}^{n} d_{il}x_l(t) + g_i(t, x(t))$$

$$g_i(t, \mathbf{x}(t)) = \sum_{l=1}^{n}(a_{il}(t) - a_{il}(T))x_l(t) -$$

$$-\sum_{l=1}^{n} K_i(T, \zeta_i(T), \eta)(T - \gamma_i(T))(x_l(t) - s_l(r))/(ms_l(r)) +$$

$$+(\chi_i(t) - K_i(T, \zeta_i(T), \eta)(T - \gamma_i(T))) - [F_i(s_1(r), \ldots, s_n(r), \mathbf{u}(T), T) -$$

$$-F_i(0, s_2(r), \ldots, s_n(r), \mathbf{u}(T), T)](x_1(t) - s_1(r))/s_1(r) - \ldots - [F_i(0, \ldots, 0, s_n(r), \mathbf{u}(T), T) -$$

$$-F_i(0, \ldots, 0, 0, \mathbf{u}(T), T)](x_n(t) - s_n(r))/s_n(r) + [F_i(\mathbf{x}(t), \mathbf{u}(t), t) - F_i(\mathbf{s}(r), \mathbf{u}(T), T)]$$

When $t \geqslant T$ the system of equations (1) can be represented in the form

$$d\mathbf{x} / dt = D\mathbf{x} + \mathbf{G}(t, \mathbf{x}(t)), \quad \mathbf{G}(t, \mathbf{x}(t)) = \mathrm{col}(g_1(t, \mathbf{x}(t)), \ldots, g_n(t, \mathbf{x}(t)))$$

Repeating the discussion employed in proving Theorem 1, we complete the proof of Theorem 2.

Consider the following model example

$$\frac{dx}{dt} = -x(t) + \int_{t-H}^{t} x^{1+\varepsilon}(\tau)\mathrm{sgn}(x(\tau))d\tau, \quad \varepsilon > 0 \tag{7}$$

It can be seen that $d(T)$ can take one of the following values: $-1 + \eta^{1+\varepsilon}Hr^{-1}, -1 - \eta^{1+\varepsilon}Hr^{-1}$. In both cases $\Lambda(d(T)) < -1 + | \eta^{1+\varepsilon}Hr^{-1} |$. Since by construction $\eta < r$, we have $\Lambda(d(T)) < -1 + | \eta^{\varepsilon}H |$ and it remains less than $-\alpha(\alpha = \text{const} > 0)$ for any finite H and sufficiently small values of r. Consequently, the solution of Eq. (7) is asymptotically stable.

As another class we will consider equations of the form

$$\frac{dx}{dt} = a(t)x(t) + F(x(t), \int_0^t K(t, s, x(s))ds) \tag{8}$$

where $F(x, y)$ is a continuous function of both variables.

For simplicity we will assume that Eq. (8) is scalar. It can be seen from further calculations that the conditions of stability of the solution of Eq. (8) derived below can be extended to similar systems of integro-differential equations.

We will impose the following conditions on the function $F(x, y)$: (1) $F(-x, y) = -F(x, y)$, (2) $F(0, y) \equiv 0$.

Suppose r is a fairly small positive number. We will denote by $s_1(r)$ points lying on the sphere surface $S(0, r)$ in the space R_1, i.e. $s_1(r) = \pm r$. We establish a correspondence between a certain number $\zeta_1(t) \in (0, t)$ and each value of t. Suppose η_1, η_2 are arbitrary numbers situated in the interval $(-r, r)$, where $\eta_1, \eta_2 \geqslant 0$ when $s_1(r) = r$ and $\eta_1, \eta_2 \leqslant 0$ when $s_1(r) = -r$. We introduce the following notation

$$d(T) = \begin{cases} a(T) + F(\eta_1, TK(T, \zeta_1, \eta_2))/\eta_1, & |\eta_2| \leq |\eta_1|, \ \eta_1 \neq 0 \\ a(T), & \eta_1 = 0 \end{cases}$$

Theorem 3. Suppose r is an arbitrary positive fairly small number. If for any $T(0 < T < \infty)$ and arbitrary $\zeta(T) \in (0, T), \eta_1, \eta_2 \in S(0, r)(\eta_1 \cdot \eta_2 \geqslant 0, | \eta_2 | \leqslant | \eta_1 |)$ the condition $d(T) < 0(d(T) < -\alpha, \alpha > 0)$ is satisfied, the solution of Eq. (8) is stable (asymptotically stable).

Proof. We will assume, to fix our ideas, that $x(0) = x_0 > 0$. Suppose that at the instant of time T the trajectory of the solution of Eq. (8) leaves the sphere $S(0, r)$, i.e. it passes through the point r on the OX axis. The trajectory cannot pass through the point $-r$ since $x_0 > 0$, while by condition (2), imposed on the function $F(x, y)$, a trajectory beginning in the segment $[0, r]$ cannot transfer into the interval $[-r, 0)$.

When $t \geqslant T$ we will represent Eq. (8) in the form

$$\frac{dx}{dt} = d(T)x(t) + (a(t) - a(T))x(t) + F(r, \psi(T)) - F(r, TK(T, \zeta_1\eta_2))x(t)/r +$$

$$+F(x(t), \psi(t)) - F(r, \psi(T)), \quad \psi(t) = \int_0^t K(t, s, x(s))ds$$

Repeating the discussion carried out when proving Theorem 1, and taking into account the fact that the trajectory of Eq. (8) can lie only in the segment [0, r], we can show that the theorem is true.

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